

Optimal Design of a Passive Vibration Absorber for a Truss Beam

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The selection of the design parameters of passive vibration absorbers attached to a long cantilevered beam is studied. This study was motivated by the need for conducting parametric analysis of dynamics and control for Space Shuttle-attached long beams. An optimization scheme using a quadratic cost function is introduced yielding the optimal sizing of the tip vibration absorber. Analytical solutions for an optimal absorber are presented for the case of one beam vibrational mode coupled with the absorber dynamics.

Nomenclature

A	= state matrix
c	= generalized damping matrix
c_c	= critical damping
c_v	= damping coefficient
$E(J)$	= expectation of the cost function J
$\bar{E}(J)$	= normalized expectation of the cost function J
EI	= bending rigidity of the beam
f	= natural frequency ratio
g	= system frequency ratio
J	= quadratic cost function
k	= generalized stiffness matrix
k_s	= spring constant
L	= Lagrangian of the system
L_d	= Lagrangian density of the beam
L_0	= Lagrangian of the vibration absorber
l	= length of the beam
m	= generalized mass matrix
m_c	= control mass
m_t	= tip mass
n	= number of beam modes
P	= solution matrix of a Lyapunov equation
Q, \bar{Q}	= weighting matrices
q	= generalized coordinates
S	= domain of parameter vector β
T	= kinetic energy of the system
t	= time
t_f	= terminal time
t_0	= initial time
V	= potential energy of the system
W_{nc}	= dissipative work energy
x	= axis along the beam, $0 \leq x \leq l$
y	= bending displacement
y_l	= bending displacement at the tip
Z^0	= second-order moment matrix of the vector z_0
z	= state vector
z_0	= initial state vector
β	= parameter vector (m_c, c_v, k_s)
η	= control mass displacement relative to tip end of the beam
λ	= overall system eigenvalue
μ_c	= equivalent control mass ratio
μ_t	= equivalent tip mass ratio
ρ	= mass density of the beam per unit length

σ_i	= frequency constant for the i th mode
ϕ_i	= assumed mode shape for the beam ($i = 1, 2, \dots$)
ω_i	= natural frequency of the beam ($i = 1, 2, \dots$) with or without tip mass
ω_s	= natural frequency of the vibration absorber

Introduction

IN recent years, passive damping has been of growing interest to the space vehicle designer because it has proven success and reliability in many other areas such as ground vehicle design. Although the investigation of optimal tuning for a vibration absorber has received attention for many years,¹⁻³ solutions are not available for models of practical systems with multiple modes and damper dynamics included. This paper presents a technique for formulating expressions of the optimal tuning law for an elastic system including a truss beam and a tip vibration absorber.

All tuning problems are simple optimization problems in which it is desirable to minimize an error criterion by selection of parameters in a model. For example, the error criterion can be the resonant peak of the deflection excited by a sinusoidal input for a two-dimensional system as shown in Refs. 2 and 3. The objective of the absorber design then is to bring the resonant amplitude down to its lowest possible value. Missing from the approach in Refs. 2 and 3, however, is the appropriate integration of the passive and active control techniques. In addition, the derivation of an optimal tuning algorithm for elastic systems with multiple modes is complex. In contrast, this paper will derive an optimal tuning absorber based on the principle of least-squares which has been used extensively in an active controller design.

Because of the numerical complexity of finite element models for truss structures, the analysis of this paper is carried out on the basis of a simple model. This model^{4,5} is described by a set of partial differential equations with equivalent structural parameters such as mass density, bending rigidity, etc. The dynamical models based on this equivalent parameter approach appear to be very tractable since the functional dependence between the performance and the model parameters can be estimated to conduct parametric analyses. To illustrate the concept in this paper, a reduced model with only two degrees of freedom is studied in detail. The method produces results for free vibrations that are in good agreement with observations from other investigators.^{2,3} Since solutions are formulated in terms of analytical expressions, the decision to size the absorber does not require guesswork and/or engineering judgment. Of course, in practical cases with multiple modes, these results may not be accurate for the case of a single mode, but they may suffice and provide at least a strong basis for the first step in the design iteration loop. The sizing of the

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vibration absorber for a system with multiple modes is discussed.

This paper is based on an optimal control theory that has been applied extensively in the control community. The approach proposed here could be interpreted as a passive control analysis although it also could be implemented as an active controller if appropriate sensors and actuators are chosen and located, and their dynamics properly formulated. The physical performance limits for a particular size of the passive absorber are presented.

Formulations

Consider a system, as shown in Fig. 1, which consists of a flexible truss beam fixed at one end and a vibration absorber attached at the other end with a tip mass m_t . Given a set of initial conditions of the beam, seek a set of parameters, including the control mass m_c , the damping coefficient c_v , and the spring constant k_s , so that the deflection y_l at the tip is minimized. Only one-dimensional bending of the truss beam is considered. Denote by EI the equivalent bending rigidity⁴ and η the deflection of the absorber mass m_c .

The kinetic energy T and potential energy V expressions are

$$2T = m_c(\dot{\eta} + \dot{y}_l)^2 + m_t \dot{y}_l^2 + \int_0^l \rho \dot{y}^2 dx \quad (1)$$

$$2V = k_s \eta^2 + \int_0^l EI y_{,xx}^2 dx \quad (2)$$

where $(\dot{}) = d()/dt$ and $()_{,xx} = Gv^2()/\partial x^2$.

The Lagrangian expression is

$$L = T - V = L_0 + \int_0^l L_d dx \quad (3)$$

in which

$$2L_0 = 2L_0(\dot{\eta}, \eta, \dot{y}_l) = m_c(\dot{\eta} + \dot{y}_l)^2 + m_t \dot{y}_l^2 - k_s \eta^2 \quad (4)$$

and

$$2L_d = 2L_d(\dot{y}, y_{,xx}) = \rho \dot{y}^2 - EI y_{,xx}^2 \quad (5)$$

The generalized Hamiltonian principle is used to derive the equations of motion; i.e.,

$$\delta \int_{t_0}^{t_f} L dt + \delta W_{nc} = \int_{t_0}^{t_f} \left[\delta L_0 + \int_0^l \delta L_d dx \right] dt + \delta W_{nc} = 0 \quad (6)$$

where the virtual work of a dissipative force is given by

$$\delta W_{nc} = -c_v \dot{\eta} \delta \eta \quad (7)$$

Discretization techniques such as finite differences, finite elements, and the Rayleigh-Ritz method are appropriate to extract the characteristics of a system. Employed in this analysis is the Rayleigh-Ritz method which is based on selection of a family of shape functions satisfying geometric boundary conditions of the problem. Consider a linear combination

$$y(x, t) = \sum_{i=1}^n q_i(t) \phi_i(x) \quad (8)$$

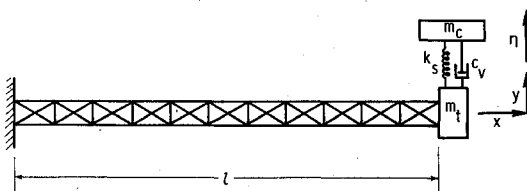


Fig. 1 Truss beam with a tip vibration absorber.

where $\phi_i(x)$ are known shape functions of the spatial coordinates linearly independent over the domain $0 \leq x \leq l$, $q_i(t)$ are unknown functions of time t , and n is a selected integer. Hence the variation of the bending displacement has the form

$$\delta y = \sum_{i=1}^n \phi_i \delta q_i \quad (9)$$

Introduction of Eq. (9) into the variational principle [Eq. (6)] leads to

$$m\ddot{q} + c\dot{q} + kq = 0 \quad (10)$$

where

$$m = \begin{pmatrix} \int_0^l \rho \phi_i \phi_j dx + (m_c + m_t) \phi_i(l) \phi_j(l) & m_c \phi_i(l) \\ m_c \phi_j(l) & m_c \end{pmatrix} \quad (11)$$

the matrix c is null, except,

$$c_{n+1, n+1} = c_v \quad (12)$$

$$k = \begin{pmatrix} \int_0^l EI \phi_{i,xx} \phi_{j,xx} dx & 0 \\ 0 & k_s \end{pmatrix} \quad (13)$$

and

$$q = (q_1, q_2, \dots, q_n, \eta)^T \quad (14)$$

General Concept of the Optimal Design for a Vibration Absorber

In adding the absorber characterized by the control mass m_c , damping coefficient c_v , and spring constant k_s (see Fig. 1), the objective is to bring the tip deflection (and/or velocity) envelope for a given initial condition to its lowest possible value. Thus we are in a position to define a cost function related to the tip deflection (and/or velocity) by which a procedure can be developed to obtain an optimal design of the absorber. One well-known technique is the least-squares error method⁶ which seeks to minimize the integral of the squared tip deflection, i.e.,

$$2J = \int_0^\infty y_l^2 dt = \int_0^\infty q^T Q q dt \quad (15)$$

where

$$Q = [\phi_1(l), \phi_2(l), \dots, \phi_n(l), 0]^T [\phi_1(l), \phi_2(l), \dots, \phi_n(l), 0] \quad (16)$$

y_l has been discretized by a series expansion as shown in Eq. (8).

Transforming Eq. (10) into a state space form yields

$$\dot{z} = Az \quad (17)$$

where

$$A = \begin{pmatrix} 0 & I \\ -m^{-1}k & -m^{-1}c \end{pmatrix}; \quad z = [q, \dot{q}]^T \quad (18)$$

The solution matrix P of the following Lyapunov equation

$$A^T P + P A = -\tilde{Q}; \quad \tilde{Q} = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix} \quad (19)$$

is given as⁶

$$P = - \int_0^\infty e^{A^T t} \tilde{Q} e^{A t} dt \quad (20)$$

The solution exists and is unique as long as the eigenvalues of A have negative real parts which are guaranteed when the damping coefficient c_v is nonzero and positive.

The cost function J in Eq. (15) becomes

$$2J = \int_0^\infty z^T \tilde{Q} z dt = z_0^T \int_0^\infty e^{A^T t} \tilde{Q} e^{A t} dt z_0 = -z_0^T P z_0 \quad (21)$$

where $z_0 = z(t=0)$. The determination of an optimal design for a vibration absorber herein obviously requires the solution of the Lyapunov equation (19) for a $(2n+2) \times (2n+2)$ matrix. If the solution matrix P is correspondingly partitioned into four $(n+1) \times (n+1)$ matrices, as shown in Eq. (19),

$$P^T = P = \begin{pmatrix} p_{qq} & p_{q\dot{q}} \\ p_{q\dot{q}}^T & p_{\dot{q}\dot{q}} \end{pmatrix} \quad (22)$$

Eq. (19) becomes

$$-\tilde{k}^T p_{q\dot{q}}^T - p_{q\dot{q}} \tilde{k} = \tilde{Q} \quad (23a)$$

$$p_{qq} - \tilde{c}^T p_{q\dot{q}} - p_{q\dot{q}} \tilde{c} = 0 \quad (23b)$$

$$p_{q\dot{q}} + p_{q\dot{q}}^T - \tilde{c}^T p_{\dot{q}\dot{q}} - p_{\dot{q}\dot{q}} \tilde{c} = 0 \quad (23c)$$

where

$$\tilde{k} = m^{-1}k; \quad \tilde{c} = m^{-1}c \quad (24)$$

Since p is a symmetric matrix, the solution of p_{qq} must be chosen such that

$$p_{qq} = p_{qq}^T \quad \text{and} \quad p_{q\dot{q}} = p_{\dot{q}q}^T \quad (25)$$

The numerical solution of the partitioned Lyapunov equation can be developed using the procedure shown in Ref. 7.

The cost function equation (21) depends on the initial condition z_0 , the damping coefficient c_v , the control mass m_c , and the spring constant k_s of the vibration absorber, and the beam characteristics. For given initial conditions z_0 and the beam characteristics, the optimal cost function thus will be a function of the vector $\beta = (m_c, c_v, k_s)_t$, representing the characteristics of the vibration absorber. Let $\beta_b = (m_{cb}, c_{vb}, k_{sb})$ be a point in the domain $S = (M_c, C_v, K_s)$ with constraints

$$m_c \in M_c, c_v \in C_v, \text{ and } k_s \in K_s \quad (26)$$

where M_c , C_v , and K_s are subspaces which restrict m_c , c_v , and k_s to lie on a practical region, such that

$$J(\beta_b) \leq J(\beta) \quad (27)$$

for every $\beta \in S$. J has an absolute minimum at β_b . The value

$$J(\beta_b) = \min\{J(\beta) : \beta \in S\} \quad (28)$$

is the minimum of J on S . With this background a criterion thus can be established relating the minimum value $J(\beta_b)$ to the optimal tuning of a vibration absorber characterized by m_c , c_v , and k_s .

The optimal design of a vibration absorber in a given realistic domain $S = (M_c, C_v, K_s)$ will be defined at the point where the absolute minimum of the optimal cost function as shown in Eq. (21) occurs.

If the initial state z_0 of the system cannot be known exactly, we may consider the initial state as a random variable with the second-order moment matrix.^{6,8}

$$E(z_0 z_0^T) = Z^0 \quad (29)$$

Taking expectation of Eq. (21) and using Eq. (29), one obtains

$$2E(J) = -\text{tr}(PZ^0) \quad (30)$$

where $\text{tr}(\cdot)$ means the trace of the matrix \cdot . The cost function becomes $E(J)$ and the criterion stated above applies.

Optimal Design of a Vibration Absorber with Single Mode Shape of the Beam

To illustrate the design concepts described previously, consider the case where only one admissible mode shape for a cantilever beam is used to discretize the system (see Fig. 1). For this case, analytical solutions of the Lyapunov solution, P in Eq. (23), are given in the Appendix.

Let us consider two cases for the optimal design with different initial conditions which are generally encountered in real systems.

Free Vibration with Initial Beam Tip Deflection

The normalized second-order moment matrix Z^0 for this case becomes

$$Z^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (31)$$

and the cost function (30) then can be written as

$$2E(J) = -p_{qq,11} \quad (32)$$

where $p_{qq,11}$ is given in the Appendix. Differentiate this equation with respect to the parameter vector β , thus finding slopes relative to c_v and k_s . Equating the slopes for c_v and k_s to zero leads to the following dimensionless equations:

$$(c_v/c_c)^2 = \mu_c(1 + \mu_t)/4(1 + \mu_t + \mu_c)^3 \quad (33)$$

$$f^2 = (1 + \mu_t)/(1 + \mu_t + \mu_c)^2 \quad (34)$$

where

$\mu_t = m_t \phi_l^2$ = equivalent tip mass ratio = tip mass/equivalent beam mass

$\mu_c = m_c \phi_l^2$ = equivalent control mass ratio = control mass/equivalent beam mass

$f = \omega_s/\omega_l$ = frequency ratio

ω_l = natural frequency of the beam without tip mass

ω_s = natural frequency of the vibration absorber

$c_c = 2m_c \omega_l$ = critical damping

A nonzero minimum of the slope relative to m_c occurs at the upper bound M_c . This bound generally depends on the type of force input and its amplitude.^{2,3}

For the case where $m_t = 0$, Eqs. (33) and (34) are reduced to

$$(c_v/c_c)^2 = \mu_c/4(1 + \mu_c)^3 \quad (35)$$

$$f = 1/(1 + \mu_c) \quad (36)$$

Equation (36) is a known result as shown in Refs. 2 and 3. The form of Eq. (35) is also identical to that in Refs. 2 and 3 with slight differences in coefficients. Figure 2 illustrates the design curve of damping constants required for most optimal operation of the vibration absorber. For an absorber with $\mu_c < 0.5$, the damping c_v/c_c increases rapidly with increasing absorber mass. The peak value of $c_v/c_c = 0.193$ occurs at $\mu_c = 0.5$. Figure 3 shows the optimal tuning results for the frequency ratio f of each absorber. For a very small absorber mass ($\mu_c \approx 0$), the vibration absorber frequency is about the same

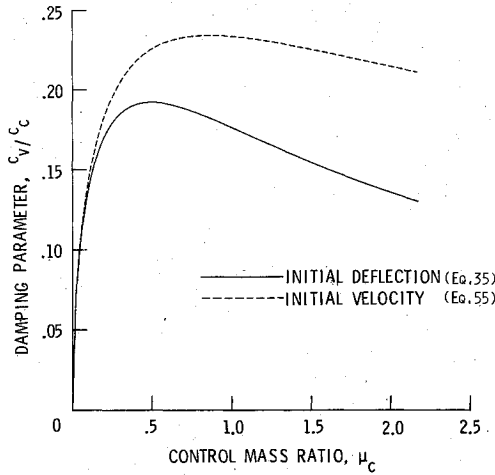


Fig. 2 Optimal tuned damper for free vibrations.

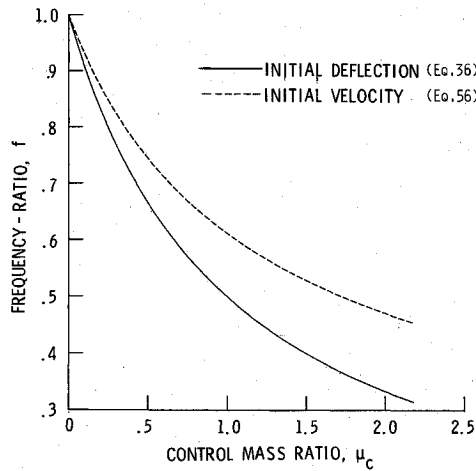


Fig. 3 Optimal tuned frequency for free vibrations.

as the beam frequency. For an absorber mass one-tenth as large as the equivalent beam mass, the absorber frequency has to be made 9% less than the beam frequency.

Substitution of Eqs. (35) and (36) into equations of motion and a short calculation yield the eigenvalue equation as follows:

$$\hat{\lambda}^4 + \sqrt{\mu_c} \hat{\lambda}^3 + (2 + \mu_c) \hat{\lambda}^2 + \sqrt{\mu_c} \hat{\lambda} + 1 = 0 \quad (37)$$

where

$$\lambda = \omega_l \hat{\lambda} / \sqrt{1 + \mu_c} \quad (38)$$

is the eigenvalue of the system. Let the four roots of this equation be λ_1 , λ_1^* , λ_2 , and λ_2^* , where $(\)^*$ means complex conjugate of $(\)$. It is seen that all of these roots are functions of μ_c only. To solve Eq. (37), reduce the fourth-degree symmetrical equation to a quadratic equation. From the equation thus obtained λ/ω_l can be written as follows with the aid of Eq. (38),

$$g_1 = \lambda_1/\omega_l = \left[-\sqrt{1-f} + \sqrt{f} + i(-\sqrt{3(1-f)} + \sqrt{f}) \right] / 4 \quad (39)$$

and

$$g_2 = \lambda_2/\omega_l = \left[-\sqrt{1-f} - \sqrt{f} + i(\sqrt{3(1-f)} + \sqrt{f}) \right] / 4 \quad (40)$$

where

$$\hat{f} = 2\sqrt{(1+f)^2 + 12f^2} - (1+7f) \quad (41)$$

$$\bar{f} = 2\sqrt{(1+f)^2 + 12f^2} + (1+7f) \quad (42)$$

The real part of λ means the decay rate, while the imaginary part indicates the oscillatory frequency. Figures 4 and 5 show plots of the eigenvalue ratios g_1 and g_2 as a function of the mass ratio μ_c . These figures obviously describe eigenvalue characteristics of the system including the beam and an optimal absorber. It is instructive to verify this result for several particular cases and to see if it reduces to known results. For $\mu_c \rightarrow \infty$, the control mass is virtually clamped and we have a single-degree system. The curve g_2 approaches a fixed value which is

$$g_2 = (-1 \pm i\sqrt{3})/2 \quad (43)$$

while g_1 becomes diminished. For a small μ_c , both system eigenvalues λ_1 and λ_2 approach the beam natural frequency ω_l . At approximately $\mu_c = 0.8$ the magnitude of the real part of g_1 (see Fig. 4) becomes a maximum.

To determine how the cost varies with the mass ratio μ_c , Eqs. (35) and (36) are substituted into Eq. (32) using the expression for $p_{qq,11}$ from the Appendix. The resulting formula for the normalized cost function with $m_t = 0$ is

$$\tilde{E}(J) = (\phi_l^2/2\omega_l) \sqrt{1 + 1/\mu_c} \quad (44)$$

Recalling the formulation of the first mode⁹

$$\omega_l = \sigma_l^2 \sqrt{EI/\rho l^4} \quad (\sigma_l = 1.875) \quad (45)$$

$$\phi_l^2 = 4/\rho l \quad (46)$$

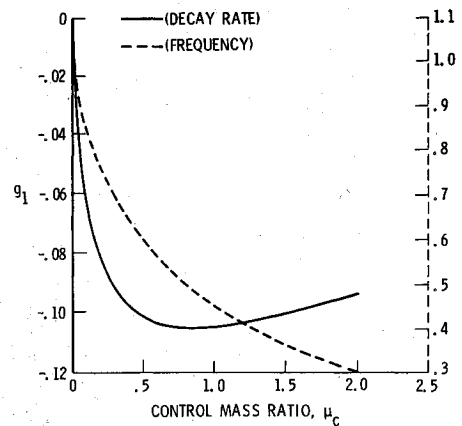


Fig. 4 Optimal system eigenvalue ratio for free vibrations.

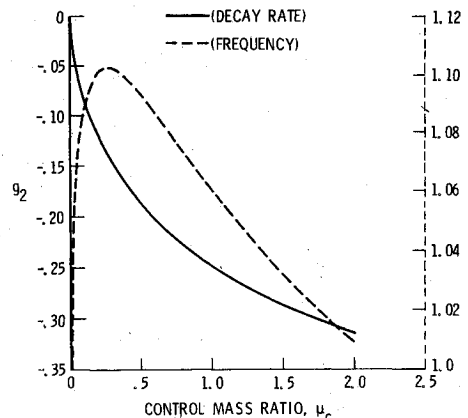


Fig. 5 Optimal system eigenvalue ratio for free vibrations.

for a cantilever beam and substituting, we find

$$\hat{l} = l/a_1 = 1/\sqrt{1 + 1/\mu_c} \quad (47)$$

where

$$a_1 = \tilde{E}(J) \sigma_1^2 \sqrt{EI\rho} / 2 \quad (48)$$

The constant a_1 generally is determined by the type of input, its possible amplitude of initial tip deflection, and beam characteristics. For the purpose of comparison, let us define a new variable

$$2E_s(J) = \phi_1^2 / \omega_1 \quad (49)$$

which is called the static normalized cost function. Equation (44) then can be written as

$$\tilde{E}(J)/E_s(J) = \sqrt{1 + 1/\mu_c} \quad (50)$$

Comparing this result with those shown in Refs. 2 and 3 it is seen that the definition of the cost function [Eq. (15)] is parallel with the physical deflection requirement under a sinusoidal force input on the vibration absorber theory given in Ref. 2. Although the type of input and the approach of designing a vibration absorber herein are completely different from that in Refs. 2 and 3, results obtained are consistent.

The curve of Fig. 6 shows the result of Eq. (47) as the solid line. It is shown that the mass ratio μ_c becomes prohibitively large as the normalized length \hat{l} approaches 1. For illustration, let us consider a large antenna feed mast described in Fig. 7. Using procedures presented in Refs. 4 and 5 for developing a simple continuum model for a truss beam, the equivalent beam properties are obtained and shown in Table 1. It is clear that the results of the simple model are in good agreement with the finite element model. Note that the equivalent bending rigidity dominates the lower modes. Substituting these beam equivalent properties for the first mode in Eq. (48), one obtains $a_1 = 1733.812 \tilde{E}(J)$. If the possible initial deflection for the generalized coordinate is $q_{10} = 5.4$ and $\tilde{E}(J) = 0.035$, the mass ratio of the absorber becomes prohibitively large as the beam length approaches 60 m. The initial displacement at the tip for the beam in physical coordinates is 3 m. The determination of the cost function J should depend on the specification of the mission requirement, such as pointing accuracy and dynamic response of the tip. Examination of Eq. (15) reveals that the cost function is proportional to the strain energy stored in the system during the course of vibration. This point of view may help determine the constraint on the cost function. Further examination of results also reveals that,

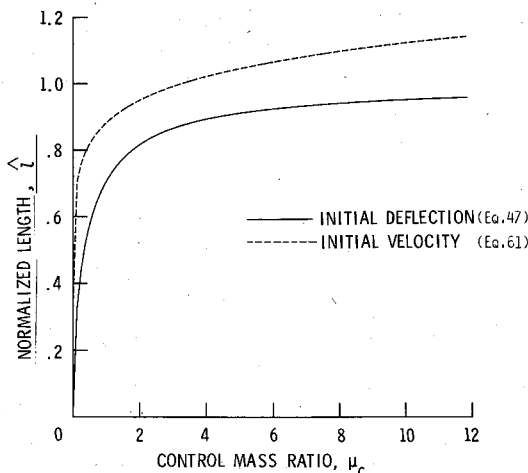


Fig. 6 Vibration absorber mass required for optimal tuned operation.

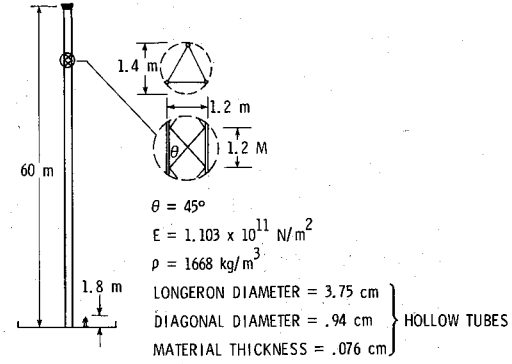


Fig. 7 Hypothetical antenna mast geometry and material properties.

Table 1 Finite element and simple model solution for the mast truss beam

Mode		Frequency	
		Finite element, rad/s	Simple model, rad/s
Bending	1	1.937	1.929
Bending	2	12.09	12.05
Bending	3	33.58	33.55
Torsion	4	48.85	49.55
Bending	5	64.65	62.25
Bending	6	103.0	106.81
Axial	7	122.6	122.5
Bending	8	140.8	157.7

Equivalent rigidities for the simple model:

Bending rigidity, EI	$1.9674 \times 10^6 \text{ N-m}^2$
Shear rigidity, GA	$2.55 \times 10^6 \text{ N}$
Torsional rigidity, GJ	$4.70 \times 10^5 \text{ N-m}^2$
Elongation rigidity, EA	$1.093 \times 10^7 \text{ N}$
Mass density, ρ	0.4945 kg-m
Inertia density, mI	0.0604 kg-m
Torsional inertia density, mJ	0.135 kg-m

for a given beam cross section and set of initial conditions, the mass ratio and beam length are related exponentially. In other words, there may not be a reasonable optimal mass ratio when the length approaches a certain value.

Free Vibration with Initial Beam Tip Velocity

The normalized second-order moment matrix Z^0 for this case is

$$Z^0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\phi_1 \\ 0 & 0 & -\phi_1 & \phi_1^2 \end{pmatrix} \quad (51)$$

and the cost function [Eq. (30)] becomes

$$2E(J) = -[p_{\dot{q}\dot{q},11} - 2\phi_1 p_{\dot{q}\dot{q},1\eta} + \phi_1^2 p_{\dot{q}\dot{q},\eta\eta}] \quad (52)$$

The same procedure applied to this case for zero slope gives

$$(c_v/c_c)^2 = \mu_c [4(1 + \mu_t) + 3\mu_c] / [16(1 + \mu_t + \mu_c)^3] \quad (53)$$

$$f^2 = [2(1 + \mu_t) + \mu_c] / [2(1 + \mu_t + \mu_c)^2] \quad (54)$$

For $\mu_t = 0$ (no tip mass),

$$(c_v/c_c)^2 = \mu_c (4 + 3\mu_c) / [16(1 + \mu_c)^3] \quad (55)$$

$$f^2 = (2 + \mu_c) / [2(1 + \mu_c)^2] \quad (56)$$

Figures 2 and 3 also illustrate design curves developed from Eqs. (55) and (56) as dashed lines. For an optimal absorber with $\mu_c < 0.87$, the tuning c_v/c_c increases rapidly with increasing μ_c . The peak value of $c_v/c_c = 0.235$ occurs at $\mu_c = 0.87$. For a very small absorber mass ($\mu_c \approx 0$), the optimal tuning gives $f \approx 1$, i.e., the absorber frequency should be the same as the beam frequency. For an absorber mass one-tenth as large as the beam equivalent mass, the absorber frequency has to be made 7% less than the beam frequency.

Substituting Eqs. (55) and (56) into equations of motion gives the following eigenvalue equation:

$$\lambda^4 + \sqrt{\mu_c(4+3\mu_c)}/4\lambda^3 + (4+3\mu_c)\lambda^2/2 + \sqrt{\mu_c(4+3\mu_c)}/4\lambda + (2+\mu_c)/2 = 0 \quad (57)$$

where

$$\lambda = \lambda\omega_1/\sqrt{1+\mu_c} \quad (58)$$

This is a quartic equation in λ , giving four roots. To solve this equation analytically is very time consuming. Let us examine the case for very large μ_c . Three eigenvalues in Eq. (57) will approach zero, while the remaining eigenvalue becomes approximately

$$\lambda/\omega_1 = -\sqrt{3\mu_c}/2 \quad (59)$$

This happens for the optimal damping coefficient $c_v \rightarrow \infty$ [see Eq. (53)] for given beam properties. It can be physically understood that when $c_v \rightarrow \infty$ the tip end of the beam is virtually clamped yielding no vibrational motion. With this background, it is intuitively seen that when $\mu_c \rightarrow \infty$ the cost function $\bar{E}(J)$ becomes zero.

Substituting Eqs. (55) and (56) into Eq. (52) gives

$$\bar{E}(J) = (\phi_1^2/4\omega_1^3)\sqrt{(4+3\mu_c)/(1+\mu_c)\mu_c} \quad (60)$$

or, with the aid of Eqs. (45) and (46),

$$\bar{l} = l/a_1 = [(1+\mu_c)\mu_c/(4+3\mu_c)]^{1/10} \quad (61)$$

where

$$a_1 = [\sigma_1^6(EI)^{3/2}E(J)/\rho^{1/2}]^{1/5} \quad (62)$$

Obviously this constant depends on the type of input, the amplitude of initial velocity, and beam properties. The result of Eq. (61) is also illustrated in Fig. 6. As in the previous case, the mass ratio μ_c becomes prohibitively large as the normalized length \bar{l} is larger than 1. For the purpose of illustration, using equivalent beam properties and $\bar{E}(J)$ as shown in the previous case gives $a_1 = 176.3407 (\bar{E}(J))^{1/5}$. In this case, the mass ratio of the absorber becomes prohibitively large as the beam length becomes larger than 90 m, in contrast to 60 m for the case of free vibration with initial deflection. Examination of Figs. 2, 3, and 6 clearly reveals that optimal parameters depend on the type of initial condition, particularly when mass ratio becomes significant.

Optimal Design of a Vibration Absorber with Multiple Modes

When the number of beam modes becomes greater than 1, no essential new aspects enter into the problem. The optimization problem of Eq. (15) represents a nonlinear optimization. Searching a minimum can be made by using the globally convergent extension of the damped Newton method as presented in Ref. 10. Several cases have been examined. For the case of free vibration where the initial condition is dominated by the first mode, the absorber sizing is determined primarily by the first mode no matter how many modes are included in the system. Although all of the test results are consistent with

engineering intuition, more studies should be made for the numerical implementation of this technique.

Concluding Remarks

The main formula shown herein relates the beam parameters to the size of the tip absorber and constitutes the basis for the optimal design of an absorber and evaluation of its performance. The optimization algorithm is based on the principle of least-squares that minimizes a quadratic cost function of the tip deflection over the parameters of tip absorber. Analytical expressions describing the change in frequency and damping for the single mode case are established yielding insight into the design picture. The attractiveness of these expressions is that they are in good agreement with expressions derived by other investigators with completely different approaches and force inputs.

The design of passive damping devices depends essentially on system parameters, and strongly on the type of inputs when the ratio of the mass of the device to that of the beam becomes significantly large. Note that large mass ratios cause difficulties in maneuvering the structure.

From various cases studied thus far it appears that damping augmentation by an absorber with a reasonable mass ratio in a structure can be successfully tuned by the present theory with a decay rate of 10%, or even 20%, for the dominant mode. However, noncolocated active control and/or distributed passive control may be necessary when multiple system modes are likely to be excited by disturbances or when strict initial conditions require prohibitively large mass ratio for the sizing of an absorber.

Appendix

For a single-mode shape, Eqs. (11-13) can be reduced to the following forms:

$$m = \begin{pmatrix} 1 + (m_t + m_c)\phi_1^2(l) & m_c\phi_1(l) \\ m_c\phi_1(l) & m_c \end{pmatrix} \quad (A1)$$

$$c = \begin{pmatrix} 0 & 0 \\ 0 & c_v \end{pmatrix} \quad (A2)$$

and

$$k = \begin{pmatrix} \omega_1^2 & 0 \\ 0 & k_s \end{pmatrix} \quad (A3)$$

where ω_1 and ϕ_1 are the frequency and corresponding mode shape, respectively, which are chosen to discretize the system. Starting from Eqs. (23-25) and substituting Eqs. (A1-A3) into them, one can solve the Lyapunov equation. This is a long and tedious job that leads to

$$p_{q\dot{q}} = [p_{q\dot{q},ij}]; i, j = 1, \eta$$

$$p_{q\dot{q},11} = 0 \quad (A4)$$

$$p_{q\dot{q},1\eta} = \phi_1(1 + m_t\phi_1^2)/2\omega_1^2 \quad (A5)$$

$$p_{q\dot{q},\eta 1} = -\omega_s^2\phi_1[1 + (m_t + m_c)\phi_1^2]^2/2\omega_1^4 \quad (A6)$$

$$p_{q\dot{q},\eta\eta} = -m_c\omega_s^2\phi_1^2[(1 + m_t + m_c)\phi_1^2]/2\omega_1^4 \quad (A7)$$

$$p_{q\ddot{q}} = [p_{q\ddot{q},ij}]; i, j = 1, \eta$$

$$\begin{aligned} p_{q\ddot{q},11} = & -\{(c_v/m_c)^2[1 + (m_t + m_c)\phi_1^2]^3 \\ & + m_c\omega_s^2\phi_1^2[1 + (m_t + m_c)\phi_1^2]^2 \\ & + \omega_1^{-2}\{\omega_s^2[1 + (m_t + m_c)\phi_1^2]^2 \\ & - \omega_1^2(1 + m_t\phi_1^2)\}^2\}/2c_v\omega_1^4 \end{aligned} \quad (A8)$$

$$\begin{aligned}
 p_{\dot{q}\dot{q},1\eta} = & -\phi_1 \left\{ (c_v^2/m_c) [1 + (m_t + m_c) \phi_1^2]^2 \right. \\
 & + m_c^2 \omega_s^2 \phi_1^2 [1 + (m_t + m_c) \phi_1^2] + (m_c \omega_s^2 / \omega_1^2) \\
 & \times \{ -\omega_1^2 (1 + m_t \phi_1^2) + \omega_s^2 [1 + (m_t + m_c) \phi_1^2]^2 \} \\
 & \times [1 + (m_t + m_c) \phi_1^2] \} / 2c_v \omega_1^4 \quad (A9)
 \end{aligned}$$

$$\begin{aligned}
 p_{\dot{q}\dot{q},\eta\eta} = & -\phi_1^2 \{ c_v^2 [1 + (m_t + m_c) \phi_1^2] + m_c^2 \omega_s^2 \\
 & \times [1 + (m_t + m_c) \phi_1^2] + (m_c \omega_s / \omega_1)^2 \{ -\omega_1^2 (1 + m_t \phi_1^2) \\
 & + \omega_s^2 [1 + (m_t + m_c) \phi_1^2]^2 \} \} / (2c_v \omega_1^4) \quad (A10)
 \end{aligned}$$

$$p_{qq} = [p_{qq,ij}]; i, j = 1, \eta$$

$$\begin{aligned}
 p_{qq,11} = & -\{ (c_v/m_c)^2 [1 + (m_t + m_c) \phi_1^2]^2 \\
 & + m_c \omega_s^2 \phi_1^2 [1 + (m_t + m_c) \phi_1^2] \\
 & + \omega_1^{-2} \{ \omega_s^2 [1 + (m_t + m_c) \phi_1^2]^2 - \omega_1^2 (1 + m_t \phi_1^2) \} \\
 & \times \{ \omega_s^2 [1 + (m_t + m_c) \phi_1^2] - \omega_1^2 \} \} / 2c_v \omega_1^2 \quad (A11)
 \end{aligned}$$

$$\begin{aligned}
 p_{qq,1\eta} = & -m_c \omega_s^2 \phi_1 \{ \omega_s^2 [1 + (m_t + m_c) \phi_1^2]^2 \\
 & - \omega_1^2 (1 + m_t \phi_1^2) \} / 2c_v \omega_1^4 \quad (A12)
 \end{aligned}$$

$$p_{qq,\eta\eta} = -m_c^2 \phi_1^2 \omega_s^4 [1 + (m_t + m_c) \phi_1^2] / 2c_v \omega_1^4 \quad (A13)$$

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